7. EXAMPLES OF CHARACTER TABLES

§ 7.1. Character Tables of Cyclic Groups

If $\theta = e^{2\pi i/n}$, and G is the cyclic group $\langle A \mid A^n \rangle$, the map $A^k \to \theta^k$ is an isomorphism and hence a faithful representation. Being a linear representation it is its own character. So we get a character χ such that $\chi(A^k) = \theta^k$. Powers of χ give all the other irreducible characters.

Example 1: $C_2 = \langle A|A^2 \rangle$

class	**1**	**A**
size	**1**	**1**
χ_1	1	1
χ_2	1	–1
order	**1**	**2**

Example 2: $C_3 = \langle A|A^3 \rangle$

class	**1**	**A**	**A^2**
size	**1**	**1**	**1**
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω
order	**1**	**3**	**3**

Example 3: $C_4 = \langle A|A^4\rangle$

class	1	A	A^2	A^3
size	1	1	1	1
χ_1	1	1	1	1
χ_2	1	i	–1	–i
χ_3	1	–1	1	–1
χ_4	1	–i	–1	i
order	1	4	2	4

Example 4: $C_6 = \langle A|A^6\rangle = 1$ Let $\theta = e^{2\pi i/6}$.

class	1	A	A^2	A^3	A^4	A^5
size	1	1	1	1	1	1
χ_1	1	1	1	1	1	1
χ_2	1	θ	θ^2	θ^3	θ^4	θ^5
χ_3	1	θ^2	θ^4	θ	θ^3	θ^4
χ_4	1	θ^3	1	θ^3	1	θ^3
χ_5	1	θ^4	θ^2	1	θ^4	θ^2
χ_6	1	θ^5	θ^4	θ^3	θ^2	θ
order	1	6	3	2	3	6

This follows the general pattern, whereby $\chi_{rs} = \theta^{(r-1)(s-1)}$.

However $C_6 \cong C_2 \times C_3$.

Let B have order 2 and C have order 3 and let A = BC, which has order 6. Taking the character tables for C_2 and C_3 we can construct the character table for C_6 as follows:

class	1 (1, 1)	A^4 (1, C)	A^2 (1, C^2)	A^3 (B, 1)	A (B, C)	A^5 (B, C^2)
size	1	1	1	1	1	1
χ_1	1	1	1	1	1	1
χ_2	1	ω	ω^2	1	ω	ω^2
χ_3	1	ω^2	ω	1	ω^2	ω
χ_4	1	1	1	-1	-1	-1
χ_5	1	ω	ω^2	-1	$-\omega$	$-\omega^2$
χ_6	1	ω^2	ω	-1	$-\omega^2$	$-\omega$
order	1	3	3	2	6	6

By observing that $\theta^2 = \omega$ and $\theta^3 = -1$ we can reconcile these tables as being the same, after suitable rearrangement.

§ 7.2. Character Tables of Abelian Groups

Example 5: Find the character table of $C_2 \times C_2 \times C_3$.

Solution: The character tables for $C_2 \times C_2$ and C_3 are respectively:

	1	A	B	AB
size	1	1	1	1
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1
order	1	2	2	2

	1	C	C^2
size	1	1	1
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω
order	1	3	3

Hence the character table for $(C_2 \times C_2) \times C_3$ is

class	11	A1	B1	AB1	1C	AC	BC	ABC
order	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	1	1	1	1
χ_2	1	–1	1	–1	1	–1	1	–1
χ_3	1	1	–1	–1	1	1	–1	–1
χ_4	1	–1	–1	1	1	–1	–1	1
χ_5	1	1	1	1	ω	ω	ω	ω
χ_6	1	–1	1	–1	ω	$-\omega$	ω	$-\omega$
χ_7	1	1	–1	–1	ω	ω	$-\omega$	$-\omega$
χ_8	1	–1	–1	1	ω	$-\omega$	$-\omega$	ω
χ_9	1	1	1	1	1	ω^2	ω^2	ω^2
χ_{10}	1	–1	1	–1	1	$-\omega^2$	ω^2	$-\omega^2$
χ_{11}	1	1	–1	–1	1	ω^2	$-\omega^2$	$-\omega^2$
χ_{12}	1	–1	–1	1	1	$-\omega^2$	$-\omega^2$	ω^2
order	1	2	2	2	3	6	6	6

class	$1C^2$	AC^2	BC^2	ABC^2
order	1	1	1	1
χ_1	1	1	1	1
χ_2	1	–1	1	–1
χ_3	1	1	–1	–1
χ_4	1	–1	–1	1

class	$1C^2$	AC^2	BC^2	ABC^2
order	1	1	1	1
χ_5	ω^2	ω^2	ω^2	ω^2
χ_6	ω^2	$-\omega^2$	ω^2	$-\omega^2$
χ_7	ω^2	ω^2	$-\omega^2$	$-\omega^2$
χ_8	ω^2	$-\omega^2$	$-\omega^2$	ω^2
χ_9	ω^2	ω	ω	ω
χ_{10}	ω^2	$-\omega$	ω	$-\omega$
χ_{11}	ω^2	ω	$-\omega$	$-\omega$
χ_{12}	ω^2	$-\omega$	$-\omega$	ω
order	3	6	6	6

§ 7.3. Character Tables of Dihedral Groups

Example 6: $\mathbf{D_8} = \langle A, B \mid A^4, B^2, BA = A^{-1}B\rangle$
The conjugacy classes are: $\{1\}$, $\{A^2\}$, $\{A, A^3\}$, $\{B, A^2B\}$, $\{AB, A^3B\}$. $H = <A^2>$ is a normal subgroup of order 2 and $G/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

class	1	A, A^3	A^2	B, A^2B	AB, A^3B	
size	1	2	1	2	2	
χ_1	1	1	1	1	1	trivial character
χ_2	1	1	1	–1	–1	from $G/\langle A^2\rangle$
χ_3	1	–1	1	1	–1	from $G/\langle A^2\rangle$

χ_4	1	–1	1	–1	1	can be induced from $G/\langle A^2\rangle$
χ_5	2	0	–2	0	0	obtained by orthogonality
order	**1**	**4**	**2**	**2**	**2**	

Example 7: $\mathbf{D_{10}} = \langle A, B \mid A^5, B^2, BA = A^{-1}B \rangle$
The conjugacy classes are: {1}, $\{A^2, A^4\}$, $\{A, A^4\}$, {B, AB, A^2B, A^3B, A^4B}. H = $<A^5>$ is a normal subgroup of order 5 and $G/H \cong \mathbb{Z}_2$.

$\Gamma_1 = \{1\}$, $\Gamma_2 = \{A, A^4\}$, $\Gamma_3 = \{A^2, A^3\}$, $\Gamma_4 = \{A^kB\}$.

class	$\mathbf{\Gamma_1}$	$\mathbf{\Gamma_2}$	$\mathbf{\Gamma_3}$	$\mathbf{\Gamma_4}$	
size	**1**	**2**	**2**	**5**	
χ_1	1	1	1	1	trivial character
χ_2	1	1	1	–1	induced $G/\langle A^5\rangle$
χ_3	2	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	0	
χ_4	2	$\frac{-1-\sqrt{5}}{2}$	$-\frac{1-\sqrt{5}}{2}$	0	orthogonality
order	**1**	**5**	**5**	**2**	

Explanation:
Since there are 4 conjugacy classes there are 4 irreducible character, and so the degrees of χ_3 and χ_4 must both be 2.

Inducing up from the subgroup $\langle A^5\rangle$ the degree 1 character:

1	**A**	$\mathbf{A^2}$	$\mathbf{A^3}$	$\mathbf{A^4}$
1	θ	θ^2	θ^3	θ^4

(where $\theta = e^{2\pi i/5}$) we get the character

class	$\mathbf{\Gamma_1}$	$\mathbf{\Gamma_2}$	$\mathbf{\Gamma_3}$	$\mathbf{\Gamma_4}$
size	**1**	**2**	**2**	**5**
$\mathbf{\Omega}$	$2 \times 1 \times 1$	$2 \times 1 \times \dfrac{\theta + \theta^4}{2}$	$2 \times 1 \times \dfrac{\theta^2 + \theta^3}{2}$	0

for D_{10}, that is:

class	$\mathbf{\Gamma_1}$	$\mathbf{\Gamma_2}$	$\mathbf{\Gamma_3}$	$\mathbf{\Gamma_4}$
size	**1**	**2**	**2**	**5**
$\mathbf{\Omega}$	2	$2\cos\dfrac{2\pi}{5}$	$2\cos\dfrac{4\pi}{5}$	0

Since $\cos\dfrac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}$ and $\cos\dfrac{4\pi}{5} = -\dfrac{1+\sqrt{5}}{4}$ we can write this as:

class	$\mathbf{\Gamma_1}$	$\mathbf{\Gamma_2}$	$\mathbf{\Gamma_3}$	$\mathbf{\Gamma_4}$
size	**1**	**2**	**2**	**5**
$\mathbf{\Omega}$	2	$\dfrac{-1+\sqrt{5}}{2}$	$-\dfrac{1+\sqrt{5}}{2}$	0

Since $\langle \Omega | \Omega \rangle = 1$, this is irreducible. We can complete the table by orthogonality.

The general dihedral group is:

$$D_{2n} = \langle A, B \mid A^n, B^2, B^{-1}AB = A^{-1}\rangle.$$

If n is odd the class equation is:

$$2n = 1 + 2 + 2 + .. + 2 + n$$

and there are $\frac{1}{2}(n + 3)$ irreducible characters: 2 linear and $\frac{1}{2}(n - 1)$ of degree 2.

If n is even the class equation is:

$2n = 1 + 1 + 2 + 2 + .. + 2 + \frac{1}{2}n + \frac{1}{2}n$

and there are $\frac{1}{2}(n + 6)$ irreducible characters: 4 linear and $\frac{1}{2}(n - 2)$ of degree 2.

§ 7.4. Character Tables of Permutation Groups

Example 7: S_3

class	**I**	**(×××)**	**(××)**	
size	**1**	**2**	**3**	
χ_1	1	1	1	trivial character
χ_2	1	1	–1	induced from G/A_3
χ_3	2	–1	0	orthogonality
order	**1**	**3**	**2**	

The permutation character is $[3, 0, 1] = \chi_1 + \chi_3$.

NOTE: S_3 is isomorphic to D_6.

Example 8: S_4

The trivial character and the permutation character are:

class	I	(××)(××)	(×××)	(××)	(××××)
size	1	3	8	6	6
χ_1	1	1	1	1	1
Π	4	0	1	2	0

Since $\langle\Pi|\Pi\rangle = 2 = 1^2 + 1^2$, Π must be the sum of two irreducible characters.

Since $\langle\Pi|\chi_1\rangle = 1$, Π must be χ_1 plus another irreducible character. So $\Pi - \chi_1$ is an irreducible character. Thus we can complete the character table for S_4.

To save space we represent a cycle structure by writing down the numbers in each cycle. (Remember that all permutations with the same cycle structure form a single conjugacy class in S_n.) So (××)(××) will be denoted by 2.2 and (××××) by 4.

class	I	2.2	3	2	4	
size	1	3	8	6	6	
χ_1	1	1	1	1	1	trivial character
χ_2	1	1	1	–1	–1	induced G/A_4
χ_3	2	2	–1	0	0	orthogonality
χ_4	3	–1	0	–1	1	$= \chi_2\chi_5$
χ_5	3	–1	0	1	–1	$\chi_5 = \Pi - \chi_1$
order	1	2	3	2	4	

Example 9: S_5

class	I	2.2	3	5	2	4	2.3
size	1	15	20	24	10	30	20
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	–1	–1	–1
χ_3	4	0	1	–1	2	0	–1
χ_4	4	0	1	–1	–2	0	1
χ_5	5	1	–1	0	1	–1	1
χ_6	5	1	–1	0	–1	1	–1
χ_7	6	–2	0	1	0	0	0
order	1	2	3	5	2	4	6

Explanation:

χ_1 is trivial and χ_2 can be obtained by inducing up from $S_5/A_5 \cong C_2$. The permutation character is:

θ_1	5	1	2	0	3	1	0

This gives $\chi_3 = \theta_1 - \chi_1$ and $\chi_4 = \chi_3\chi_2$.

The sum of squares of the degrees is 120 so the sum of squares of the remaining irreducibles is 86. A little calculation reveals that there are only two solutions to

$$a^2 + b^2 + c^2 = 86$$

for positive integers a, b, c. They are 1, 2, 9 and 5, 5, 6. The first case is impossible because it would imply another linear character.

None of its value could be 0 (they have to be roots of unity) so multiplying by χ_2 would give a fourth linear character, a contradiction.

Hence we may take χ_5, χ_6, χ_7 as being of degrees 5, 5 and 6 respectively. Clearly $\chi_7\chi_2 = \chi_7$ (it is the only irreducible of degree 6) so the last three entries in the χ_7 row must be zero and the two entries immediately above them must each total zero.

Also, corresponding entries in the rest of χ_5, χ_6 must be equal. With a fair amount of effort one can now complete the character table by orthogonality. (Remember that since all classes are their own inverses all the characters are real.) The calculations can be made somewhat easier by calculating χ_4^2:

θ_2	16	0	1	1	4	0	1

This contains χ_1 and χ_3 so let $\theta_3 = \theta_2 - \chi_1 - \chi_3$:

θ_3	11	0	–1	1	1	–1	1

Since $\langle\theta_3 \mid \theta_3\rangle = 2$, θ_3 is a degree 11 character which is the sum of two irreducible characters. It must be the sum of the degree 6 and one of the degree 5 characters, say χ_5.

Knowing that the last three entries in χ_7 are zero enables us now to easily complete the last three columns and with a little more work we complete the table.

We can save even more work by inducing the degree 2 character of S_4 up to S_5:

θ_4	10	2	–2	0	0	0	0

Neither of the degree 4 characters are present so it must be the sum of the two degree 5 characters.

Knowing $\chi_5\chi_2 = \chi_6$ and $\chi_5 + \chi_6 = \theta_4$ we can readily complete the table.

Example 10: A_5

Here the 24 5-cycles split into two classes of size 12.
To work out the class size of a permutation you work out the size of its centraliser – that is, the number of permutations that commute with it.

A typical 3-cycle (123) commutes with I , (123) and (132) as well as any permutation on {4, 5}. Moreover it commutes with any product of these. Hence the centraliser has order 3 × 2 = 6. The size of the conjugacy class is the index of the centraliser so, in this case, it is $\frac{60}{6}$ = 10. Since there are only ten 2-cycles altogether they must all be in the one conjugacy class. Repeating this for the other cycle structures we find that permutations with the same cycle structure form a single conjugacy class in A_5 *except* the 5-cycles. Here the centraliser of a 5-cycle is the cyclic group it generates, and so has order 5 and hence index 12. Since there are 24 5-cycles they must split into 2 conjugacy classes.

class	I	2.2	3	5	
size	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	–1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	–1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	–1	–1
χ_5	5	1	–1	0	0
order	1	2	3	5	5

Explanation:

χ_1 is trivial. The permutation character is:

θ_1	5	1	2	0	0

Hence we find $\chi_4 = \theta_1 - \chi_1$.

Since the sum of squares of the degrees must total 60 we deduce that the remaining irreducible characters must have degrees 3, 3 and 5.

Since $\chi_2((12)(34))$, $\chi_3((12)(34))$ and $\chi_5((12)(34))$ must be sums of ±1's and so their values must be ±1 or ±3 for χ_2 and χ_3 and ±1, ±3 or ±5 for χ_5. By considering the lengths of χ_2, χ_3 and χ_5 we discover that ±3 or ±5 are too big and a little more investigation reveals that the values are as given in the second column of the character table. The remainder of the table can be completed by some extensive orthogonality calculations.

A quicker method of completing the table uses the technique of inducing up from subgroups. Inducing up from A_4 gives:

θ_1	5	1	2	0	0
θ_2	5	1	–1	0	0
θ_3	15	–1	0	0	0

(The two non-trivial linear characters of A_4 both give θ_2 when induced up.)
$\chi_5 = \theta_2$ and $\chi_4 = \theta_1 - \chi_1$. Now θ_3 contains one copy of each of χ_4 and χ_5. Subtracting we get:

θ_4	6	–2	0	0	0

Since $\langle\theta_4 \mid \theta_4\rangle = 2$, θ_4 is the sum of two irreducible characters. Since the sum of squares of the degrees = 60, having obtained χ_1, χ_4, χ_5 we conclude that the remaining characters have degree 3 and that θ_4 is their sum. With this knowledge, and orthogonality, we find χ_2 and χ_3.

§ 7.5. Character Tables of Linear Groups

Example 11: GL(2,3) This is the group of all invertible 2×2 matrices over $\mathbb{Z}_3$. We'll denote this by G.

The first thing to do is to find out the order of G. Now the number of 2×2 matrices altogether, over $\mathbb{Z}_3$ is $3^4 = 81$. To be invertible the rows of the matrix must be linearly independent. For the first row this simply means that it has to be non-zero, so there are 8 possibilities. For each of these the second row has to be anything except the 3 multiples of the first row. This gives $|G| = 8 \times 6 = 48$.

We now need to find the conjugacy classes, or to use the language of linear algebra, the similarity classes. Brute strength would seem far too tedious, but remember that similar matrices have the same characteristic polynomial. So we'll enumerate all the possible characteristic polynomials and then find how many matrices share each such polynomial. Of course if two characteristic polynomials are different the corresponding matrices cannot be similar, but if two matrices have the same characteristic polynomial that doesn't guarantee that they are similar. Still, that seems to be a good place to start.

There are 9 monic quadratics over $\mathbb{Z}_3$, but clearly we must reject those that have zero constant term. The corresponding matrices will have a zero eigenvalue and hence cannot be invertible. So there are just 6 possible characteristic polynomials: $\lambda^2 \pm 1$and $\lambda^2 \pm \lambda \pm 1$. (It's convenient to write 2 as -1.)

Now, of course, these eigenvalues may not be in $\mathbb{Z}_3$. Some of these polynomials will be prime over $\mathbb{Z}_3$, but that doesn't matter.

Let's write the characteristic polynomial in the form $\lambda^2 - t\lambda + \Delta$, where t is the trace and Δ is the determinant. We must have $\Delta \neq 0$.

For each such polynomial we'll find the number of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose characteristic polynomial is

$$\lambda^2 - t\lambda + \Delta.$$

Since the trace is t we will have 3 choices for *a* and that will determine *d*.

Now the determinant $\Delta = ad - bc$ so that if $b \neq 0$ (2 choices) we will have $c = \dfrac{ad - \Delta}{b}$ and so 6 matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with that characteristic polynomial.

But if $b = 0$, c can be arbitrary.

But we must have $a(t - a) = \Delta$, that is, $a^2 - at + \Delta = 0$.

The discriminant of this quadratic is $D = t^2 - 4\Delta = t^2 - \Delta$, since we are working over $\mathbb{Z}_3$.

If D = 0 there is 1 solution for *a*,
if D = 1 there are 2 and
if D = 2 there are none.

These give an additional 3, 6 or 0 matrices respectively for each of the relevant characteristic polynomials.

t	**Δ**	**D**	**# with $b \neq 0$**	**# with $b = 0$**	**# matrices**	**zeros**
0	1	2	6	0	6	
0	2	1	6	6	12	1, 2
1	1	0	6	3	9	2, 2
1	2	2	6	0	6	
2	1	0	6	3	9	1, 1

2	2	2	6	0	6	

If $\chi(\lambda)$ has zeros in $\mathbb{Z}_3$ then matrices with that characteristic polynomial are similar to a Jordan Form, giving one conjugacy class if $\chi(\lambda)$ has distinct zeros and 2 conjugacy classes if it has a repeated zero (one class will consist of a single scalar matrix). Thus GL(2, 3) has 8 conjugacy classes.

A simple calculation of the centralisers of the matrices $\begin{pmatrix}0&1\\2&0\end{pmatrix}$, $\begin{pmatrix}0&1\\1&1\end{pmatrix}$, $\begin{pmatrix}1&1\\2&1\end{pmatrix}$ shows that they each have 6 conjugates.

		tr	Δ	size
Γ_1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	2	1	1
Γ_2	$\begin{pmatrix}2&0\\0&2\end{pmatrix}$	1	1	1
Γ_3	$\begin{pmatrix}0&1\\2&0\end{pmatrix}\begin{pmatrix}0&2\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}1&2\\2&2\end{pmatrix}\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}2&2\\2&1\end{pmatrix}$	0	1	6
Γ_4	$\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}0&2\\2&1\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}1&2\\2&0\end{pmatrix}\begin{pmatrix}2&1\\2&2\end{pmatrix}\begin{pmatrix}2&2\\1&2\end{pmatrix}$	1	2	6
Γ_5	$\begin{pmatrix}0&1\\1&2\end{pmatrix}\begin{pmatrix}0&2\\2&2\end{pmatrix}\begin{pmatrix}1&1\\2&1\end{pmatrix}\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}2&1\\1&0\end{pmatrix}\begin{pmatrix}2&2\\2&0\end{pmatrix}$	2	2	6
Γ_6	$\begin{pmatrix}0&1\\2&1\end{pmatrix}\begin{pmatrix}0&2\\1&1\end{pmatrix}\begin{pmatrix}1&1\\2&0\end{pmatrix}\begin{pmatrix}1&2\\1&0\end{pmatrix}\begin{pmatrix}2&0\\1&2\end{pmatrix}\begin{pmatrix}2&0\\2&2\end{pmatrix}\begin{pmatrix}2&1\\0&1\end{pmatrix}\begin{pmatrix}2&2\\0&2\end{pmatrix}$	1	1	8

Γ_7	$\begin{pmatrix}0&1\\2&2\end{pmatrix}\begin{pmatrix}0&2\\1&2\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\2&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}$ $\begin{pmatrix}2&1\\2&0\end{pmatrix}\begin{pmatrix}2&2\\1&0\end{pmatrix}$	2	1	8
Γ_8	$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&2\\2&0\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}1&0\\1&2\end{pmatrix}\begin{pmatrix}1&0\\2&2\end{pmatrix}\begin{pmatrix}1&1\\0&2\end{pmatrix}$ $\begin{pmatrix}1&2\\0&2\end{pmatrix}\begin{pmatrix}2&0\\0&1\end{pmatrix}\begin{pmatrix}2&0\\1&1\end{pmatrix}\begin{pmatrix}2&0\\2&1\end{pmatrix}\begin{pmatrix}2&1\\0&1\end{pmatrix}\begin{pmatrix}2&2\\0&1\end{pmatrix}$	0	2	12

Now we have to start finding characters. To begin with, SL(2, 3), the group of matrices with determinant 1, will be a normal subgroup of index 2. This will give a non-trivial linear character.

Let representatives of the 8 conjugacy classes be as follows:

Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}2&0\\0&2\end{pmatrix}$	$\begin{pmatrix}0&1\\2&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&2\end{pmatrix}$	$\begin{pmatrix}0&1\\2&1\end{pmatrix}$	$\begin{pmatrix}0&1\\2&2\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	1	1	6	6	6	8	8	12
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	1	1	–1

There's just one other proper, non-trivial normal subgroup, namely the non-zero scalar matrices. These are in fact the elements of the centre of G, which is clearly a normal subgroup of order 2. The quotient group will have order 24. Perhaps G/Z(G) is isomorphic to S_4. Can we find 4 things that the elements of G permute?

They permute the non-zero row vectors by multiplication on the right. But that would give a quotient that's isomorphic to a subgroup of S_8. However the 8 non-zero row vectors form four 1-dimensional subspaces and these will be permuted by right multiplication.

Note that we don't need to fully identify which permutation corresponds to each matrix, only the number of subspaces fixed by the matrix. So our permutation representation won't distinguish between matrices that induce an (××××) permutation and those that induce a (××)(××) permutation. The value of the character will be 0 in each case (no elements fixed).

After a little work we can produce a permutation character, θ, of degree 4.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	**1**	**1**	**6**	**6**	**6**	**8**	**8**	**12**
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	1	1	–1
θ	4	4	0	0	0	1	1	2

Now $\langle\theta|\theta\rangle = \dfrac{16 + 16 + 0 + 0 + 0 + 8 + 8 + 48}{48} = 2$, so θ is the sum of two irreducibles.

Moreover $\langle\chi_1|\theta\rangle = 1$ so a new irreducible character is

$\chi_3 = \theta - \chi_1$. And since $\chi_3\chi_2 \neq \chi_3$ it must be yet another irreducible character that we can call χ_4. So now we are well on the way.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	**1**	**1**	**6**	**6**	**6**	**8**	**8**	**12**
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	1	1	–1
χ_3	3	3	–1	–1	–1	0	0	1
χ_4	3	3	–1	1	1	0	0	–1
χ_5								
χ_6								
χ_7								
χ_8								
order	1	2	4	8	8	6	3	2

We might be able to use the fact that the quotient G/Z(G) is isomorphic to S_4. But let's see what we can do by inducing up from subgroups. Even inducing up from the trivial character can be useful. But we need some nice big subgroups so that the degree of the induced character is small.

The subgroup SL(2, 3), consisting of matrices with determinant 1 is nice and big. It has index 2 and so gives a character of degree 2. Unfortunately it is just $\chi_1 + \chi_2$. The problem is that it doesn't cut across conjugacy classes. Every conjugacy class is either entirely in or entirely out and so the proportions are either 1 or 0.

But the upper triangular matrices gives a fairly large subgroup. These are those of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and they form a subgroup, U, of order 12. Likewise the lower triangular matrices form another subgroup, L, of order 12. Inducing up from these we get degree 4 characters.
Unfortunately these turn out to be $\chi_1 + \chi_3$ and $\chi_1 + \chi_3$. All we have done is to provide an alternative way of getting χ_3 and χ_4.

Let H be the cyclic subgroup generated by $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.
Then $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$, $A^4 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $A^5 = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$,
$A^6 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, $A^7 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$.
Inducing the trivial character up from H we get:

class	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	**1**	**1**	**6**	**6**	**6**	**8**	**8**	**12**
Ψ	6	6	2	2	2	0	0	0

Now $\langle\Psi|\Psi\rangle = 3$, $\langle\Psi|\chi_1\rangle = 1$ and $\langle\Psi|\chi_4\rangle = 1$.
Hence $\chi_5 = \Psi - \chi_1 - \chi_4$ is a new irreducible character of degree 2.

We're doing well.

class	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	**1**	**1**	**6**	**6**	**6**	**8**	**8**	**12**
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	1	1	–1
χ_3	3	3	–1	–1	–1	0	0	1
χ_4	3	3	–1	1	1	0	0	–1
χ_5	2	2	2	0	0	–1	–1	0
χ_6								
χ_7								
χ_8								
order	1	2	4	8	8	6	3	2

The sums of squares of the remaining three degrees must total 24 and the only solution is

$$16 + 4 + 4 = 24.$$

Also the ‘length’ of each column (sum of squares of the moduli) must total 24 divided by the respective sizes of the conjugacy classes, and this allows us to fill in a lot of the cells with zeros.

class	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	**1**	**1**	**6**	**6**	**6**	**8**	**8**	**12**
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	1	1	–1
χ_3	3	3	–1	–1	–1	0	0	1
χ_4	3	3	–1	1	1	0	0	–1
χ_5	2	2	2	0	0	–1	–1	0
χ_6	4		0					0

χ_7	2		0					0
χ_8	2		0					0
order	1	2	4	8	8	6	3	2

Now $\chi_6(\Gamma_7)$ must be real (otherwise its conjugate would be another degree 4 irreducible) and the only possible real numbers that can be made up from the set $\{1, \omega, \omega^2\}$ are:

$$1 + 1 + 1 + 1 = 4 \text{ and } 1 + 1 + \omega + \omega^2 = 1.$$

But 4 would be too big for the 'length' of the Γ_7 column to be 6. Besides if all the eigenvalues were 1 then the elements of Γ_7 and would therefore be their only conjugate. So $\chi_6(\Gamma_7) = 1$. The remaining entries in the Γ_7 column must be the sum of two numbers chosen from $\{1, \omega, \omega^2\}$. Moreover they must be real since $\Gamma_7^{-1} = \Gamma_7$. (We can see this without any matrix calculations because if it was otherwise there would be another conjugacy class where the elements have order 3.) The only possibilities are therefore $1 + 1 = 2$ (impossible the "length" of column Γ_7 would be too big) or $\omega + \omega^2 = -1$. So these entries are both -1.

If $a = \chi_6(\Gamma_2)$, $b = \chi_7(\Gamma_2)$ and $c = \chi_8(\Gamma_2)$, then by column orthogonality we get:
$4a + 2b + 2c = -24$ and $a - b - c = 0$ from which we deduce that $a = -4$ and $b + c = 4$.
Now b, c must be each one of 2, 0 or -2 and the only way we can get orthogonality with the first column is for them to both be -2.

class	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	**1**	**1**	**6**	**6**	**6**	**8**	**8**	**12**
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	1	1	–1
χ_3	3	3	–1	–1	–1	0	0	1
χ_4	3	3	–1	1	1	0	0	–1
χ_5	2	2	2	0	0	–1	–1	0
χ_6	4	–4	0				1	0
χ_7	2	–2	0				–1	0
χ_8	2	–2	0				–1	0
order	1	2	4	8	8	6	3	2

Nearly there!

If in the Γ_4 and Γ_5 columns the entries for χ_6, χ_7 and χ_8 are a, b, c respectively then by orthogonality with the 1st and 7th columns we have $4a + 2b + 2c = 0$ and $a - b - c = 0$, which gives $a = 0$ and $b + c = 0$. So we can write 0, x and $-x$ for Γ_4 and 0, y, $-y$ for Γ_5.

If a, b, c are the corresponding entries in the Γ_6 column then $4a + 2b + 2c = 0$, as before. But this time we get

$$3 + a - b - c = 0.$$

Taken together these give $a = -1$ and $b + c = 2$.

class	$\mathbf{\Gamma_1}$	$\mathbf{\Gamma_2}$	$\mathbf{\Gamma_3}$	$\mathbf{\Gamma_4}$	$\mathbf{\Gamma_5}$	$\mathbf{\Gamma_6}$	$\mathbf{\Gamma_7}$	$\mathbf{\Gamma_8}$
size	**1**	**1**	**6**	**6**	**6**	**8**	**8**	**12**
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	1	1	–1
χ_3	3	3	–1	–1	–1	0	0	1
χ_4	3	3	–1	1	1	0	0	–1
χ_5	2	2	2	0	0	–1	–1	0
χ_6	4	–4	0	0	0	–1	1	0
χ_7	2	–2	0	x	y	z	–1	0
χ_8	2	–2	0	$-x$	$-y$	2 – z	–1	0
order	1	2	4	8	8	6	3	2

Nearly there!

Now the entries in the Γ_6 column must be real because clearly $\Gamma_6^{-1} = \Gamma_6$ (it is the only conjugacy class with elements of order 6) and hence z is real. Using the "length" of the Γ_6 column we have $4 + z^2 + (2 - z)^2 = 6$ and hence z = 1. Since $\Gamma_4^{-1} = \Gamma_5$ we must have x, y being conjugates of one another and since we can't have two identical columns x, y are not real. It follows that χ_8 is the conjugate of χ_7 and so, $-x$ is the conjugate of x. This means that x is pure imaginary, as is y.

But by the "length" of the Γ_4 column, $4 + 2|x|^2 = 8$ and so $|x| = \sqrt{2}$.

Hence $x = \pm\sqrt{2}i$ and $y = \pm\sqrt{2}i$. They must have opposite signs so, without loss of generality we may take $x = \sqrt{2}i$ and $y = -\sqrt{2}i$. The completed table is now as follows:

class	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	1	1	6	6	6	8	8	12
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	1	1	–1
χ_3	3	3	–1	–1	–1	0	0	1
χ_4	3	3	–1	1	1	0	0	–1
χ_5	2	2	2	0	0	–1	–1	0
χ_6	4	–4	0	0	0	–1	1	0
χ_7	2	–2	0	$\sqrt{2}i$	$-\sqrt{2}i$	1	–1	0
χ_8	2	–2	0	$-\sqrt{2}i$	$\sqrt{2}i$	1	–1	0
order	1	2	4	8	8	6	3	2

Phew! That was hard work. But there are other things we might have done. For example, if $\theta = e^{2\pi i/8} = \dfrac{1+i}{\sqrt{2}}$ we could have induced up the character $\chi(x^r) = \theta^r$ from H to G. This would have given the character

class	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	1	1	6	6	6	8	8	12
χ^G	6	–6	0	$\sqrt{2}i$	$-\sqrt{2}i$	0	0	0

It turns out that this is $\chi_6 + \chi_7$ but we would have had some work to do to split it.

Perhaps we should have used the fact that $G/Z(G) \cong S_4$ at the very beginning. We would have had to establish this isomorphism by looking at the permutations that the matrices have on the 1-dimensional subspaces and then working out how the conjugacy classes of S_4 (there are five of them) stretch out to the eight classes for *G*. In any case this would only have given us five of the eight irreducible characters so we would still have had some work in finding the remaining three.

§ 7.6. Techniques for Constructing Character Tables

By now you will have realised that constructing character tables doesn't involve a straightforward algorithm. It's more like doing a jigsaw puzzle, or solving a Sudoko problem, only much more challenging. There are several techniques that need to be employed, usually in conjunction with one another. Each technique supplies a piece but it requires considerable experience to decide in advance which ones are best to employ at each stage. The techniques are:

(1) Use the direct sum technique if your group is a direct sum of smaller groups.

(2) Induce up from quotient groups. This will give you irreducible characters straight away. The smaller the normal subgroup, the larger the quotient group and the more irreducibles you can get in one go – provided you have the character table for the quotient. Of course this technique is useless for simple groups!

(3) Induce up from subgroups. More often than not this will give you reducible characters. You may be lucky to find irreducible characters that you have already constructed within them and so be able to subtract them off to get to an irreducible one. Here we want to use large subgroups so as to get smaller degrees. If the degree is too large the character may have too many summands to be able to split it up. At least there will always be plenty of subgroups to try, even in a simple group. Very often inducing up from the trivial character of a subgroup can be quite useful, and one doesn't need to know the character table of the subgroup for that. But sometimes it is useful to find at least some of the characters of the subgroup and to induce up from those.

(4) Find some permutation representations.

(5) Use orthogonality either row or column orthogonality. For row orthogonality don't forget to weight each product by the size of the conjugacy class. Moreover don't forget that one of the rows, or columns, must be conjugated. If one of these is real then this is the one you'll conjugate and so you can forget conjugation. However you will get a situation where things appear to have gone wrong and almost certainly it will be because both rows or both columns contain non-real entries and you will have forgotten to conjugate one of the rows or columns. This will give you some linear equations where the variables are the unknown entries. This is only useful as a rule if you have at most three variables.

(6) Use the lengths of the rows or columns (the sum of squares of the moduli of the entries). For rows these should be equal to 1 (don't forget to weight each by the size of the conjugacy class). For columns they should equal the order of the centralizer, that is the order of the group divided by the size of the conjugacy class.

(7) Identify when conjugacy classes are inverses of one another and which are their own inverses. If $\Gamma^{-1} = \Gamma$ then its column will only contain real entries. If $\Gamma_i^{-1} = \Gamma_j$ the corresponding entries in two columns will be complex conjugates of one another. Do the same with the rows. For example if there is only one irreducible with a certain degree it clearly must be its own conjugate and hence have real entries.

(8) Use the fact that the value of a character of degree *n* on a conjugacy class whose elements have order *m* must be the sum of *n* m^{th} roots of unity. This is particularly useful when the degree and order are small. For example, if $n = 2$ and $m = 2$ then the entry must be 2, 0 or –2. If $n = 2$ and $m = 3$ the entry must be 2, $1 + \omega$, $1 + \omega^2$, 2ω, $2\omega^2$ or -1 ($= \omega + \omega^2$). If you know that it is real then 2 or –1 are the only possibilities.

(9) If you can find G' then you can use the fact that $|G:G'|$ is the number of linear characters.

(10) Where a conjugacy class has size 1 the entries in that column must have the same absolute value as the degree. That is because these elements will be in the centre, and so correspond to scalar matrices in the underlying representation. All entries for conjugacy classes of size bigger than 1 must have modulus strictly smaller than the degree.

There is a list of character tables for groups whose order is at most 16 in a document under the TABLES tab on the Postgraduate page on this website. Ultimately this will extend to all groups whose order is at most 100, except for those of orders 64 and 96.